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LETTER TO THE EDITOR

Low energy potential scattering and the limit of large dimensionality

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Abstract. The large N (N = spatial dimensionality) expansion has already proven its efficacy in providing good approximation to the low energy bound states of quantal systems. The method is extended to yield easily calculable and surprisingly adequate estimates for scattering lengths for potentials which can support up to one bound state.

The method of large-N expansions has provided a useful approximation scheme and is based on the surprising fact that increasing the number of degrees of freedom simplifies the analysis and this approach has amply proven its efficacy in fields as disparate as nuclear physics, critical phenomena and particle physics (Witten 1980, Yaffe 1982, 1983, Ma 1976, Iachello 1981, 't Hooft 1974a, b). Furthermore, at a more mundane level, the $1/N$ expansion has furnished a powerful tool for solving the Schrödinger equation to obtain low lying bound states in potential problems (Mlodinow and Shatz 1982, Mlodinow and Papanicolau 1980, 1981, Gangopadhyay *et al* 1984, Ader 1983, Sukhatme *et al* 1983). The most widely used perturbative method employed for problems not admitting analytical solution requires for its success the existence of a small parameter and a convergence of the corresponding series for relevant quantities, whereas the large N-method introduces a new expansion parameter which is $1/N$. However, the method has so far been restricted to applications dealing with bound state problems whereas the present letter is an attempt to extend the large N method to scattering by providing an approximation to the estimation of scattering length.

The radial Schrödinger equation in N-dimensions for a particle moving in a potential $\hat{V}(r)$ (appropriately scaled and defined later) is, in units of $\hbar = 1 = 2m$, given by

$$\left(\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}-\frac{l(l+N-2)}{2r^2}-\hat{V}(r)+k^2\right)R_l(r)=0 \tag{1}$$

which may be transformed into the effective one-dimensional form (in a semi-infinite region $0 \leqslant r \leqslant \infty$) of the Schrödinger equation through the substitution,

$$R_l(r) = r^{(N-1)/2}u_l(r)$$

to yield, for the s wave ($l=0$) states the equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-\frac{(N-3)(N-1)}{4r^2}-\hat{V}(r)\right)u(r)=0. \tag{2}$$

For potentials less singular at the origin than $1/r^2$ the radial function $u(r)$ behaves as $r^{(N-1)/2}$ for small r. Accordingly if the potential $\hat{V}(r)$ is short ranged, the product $\hat{V}(r)u(r)$ occurring in equation (2) shall generally possess a maximum, which, for large N and appropriately scaled $\hat{V}(r)$, shall become sharper and sharper enabling, in leading order, the approximate replacement of that term in the Schrödinger equation by a Dirac-delta potential located at a value of the radial coordinate $r=a$ where $\ln \hat{V}(r)+\frac{1}{2}(N-1)\ln r=f(r)$ is extremal, and has a corresponding strength given by $e^{f(a)}(2\pi/f''(a))^{1/2}$.

For the sake of illustration consider the Yukawa potential $V(r)=-V_0\,e^{-\mu r}/\mu r$. It is appropriate to consider the scaled potential $\hat{V}(r)=-V_0\,e^{-\tilde{\mu}Nr}/\tilde{\mu}Nr$ and accordingly $f(r)=-\tilde{\mu}Nr+\frac{1}{2}(N-3)\ln r$, wherein the required extremum occurs at the point $r=a=(N-3)/2\tilde{\mu}N$ and the effective pseudo-potential in the limit of large N becomes

$$-\frac{2V_0}{N\tilde{\mu}}\sqrt{\frac{\pi}{N}}\,e^{-N/2}\,\delta\left(r-\frac{1}{2\tilde{\mu}}\right),$$

and next substituting the value $N=3$ (the dimensionality of the given problem), the effective pseudo potential for three dimensions is given by

$$-\frac{2V_0}{\mu}\sqrt{\frac{\pi}{3}}\,e^{-3/2}\,\delta\left(r-\frac{3}{2\mu}\right).$$

In a similar way the effective pseudo-potential may be trivially calculated for various potentials.

For potentials with sharp boundaries such as the square well for which the derivative does not exist we take the relevant product of the wavefunction and potential to pile up at the edge ($r=a$) of the square well and to provide a δ function there. To calculate the strength 'λ' of the pseudo-potential $-\lambda\delta(r-a)$ we employ the knowledge of the exact wavefunction for the square well potential and obtain

$$\lambda=\frac{\int_0^\infty R(r)V(r)r^{N-1}\,\mathrm{d}r}{\int_0^\infty R(r)\,\delta(r-a)r^{N-1}\,\mathrm{d}r},$$

which in the low energy limit is $\alpha j_1(\alpha a)/j_0(\alpha a)$ where $\alpha^2=k^2+v_0$.

Now once the pseudo-potential $-\lambda\delta(r-a)$ is calculated, the scattering length is simply given by,

$$a_{\mathrm{sc}}=-\lambda a^2/(1-\lambda a),$$

which agrees exactly with the exact expression for the case of the square well (Schiff 1955).

The simple procedure delineated above is employed to calculate the scattering length in units of the parameter of various potentials as a function of depth (or strength) of the interaction and the results depicted graphically. For the sake of comparison, the scattering length in the Born approximation are also shown. The surprising adequacy of the approximation developed here is quite apparent. In particular the qualitative feature of the divergence of the scattering length as the strength becomes sufficient to support one bound-state and the concommitant change in signs of the scattering length as the strength increases further is well reproduced in the scheme considered here.

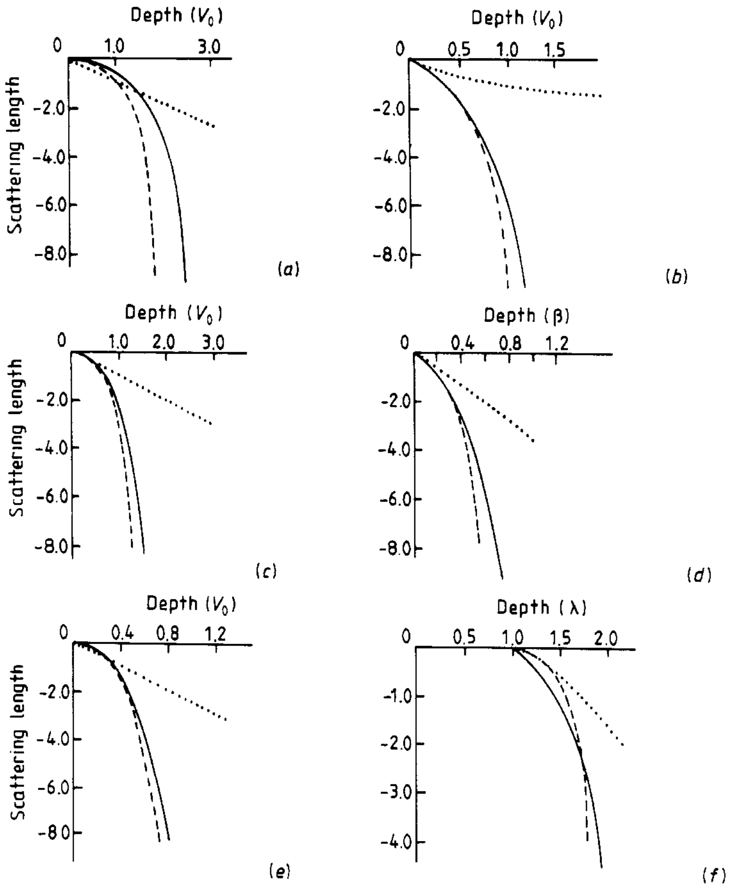

Figure 1. Scattering length, in units where $\mu = 1$ as a function of potential depth for various potentials. Exact values are displayed by full curves, while the Born and large N approximates are shown by dotted and broken curves respectively. (a) Gaussian, $V(r) = -V_0\,e^{-\mu r^2}$; ($b$) Exponential, $V(r) = -V_0\,e^{-\mu r}$; (c) Yukawa, $V(r) = -(V_0/\mu)\,e^{-\mu r}r^{-1}$; ($d$) Bargmann, $V(r) = -2\beta\mu^2\,e^{-\mu r}(1+\beta\,e^{-\mu r})^{-2}$; ($e$) Hulthen, $V(r) = -V_0\,e^{-\mu r}(1-e^{-\mu r})^{-1}$; ($f$) Poschl Tellar, $V(r) = -\mu^2\lambda(\lambda-1)\,\mathrm{sech}^2\,\mu r$.

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